The Fourier Series of Gegenbauer's Function

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1. Introduction. If N is a positive integer, the Gegenbauer polynomial C_N^ν is known to have the representation [3, vol. 2, p. 175]

$$(1.1)\qquad C_N^\nu(\cos\theta) = \sum_{m=0}^{N} \frac{(\nu)_m\,(\nu)_{N-m}}{m!\,(N-m)!}\cos(N-2m)\theta \quad ,$$

where $(\nu)_m = \Gamma(\nu+m)/\Gamma(\nu)$. The Fourier series of Gegenbauer's function $C_\alpha^\nu(\cos\theta)$ with general (possibly complex) α does not appear to have been given previously, even in the special case of Legendre's function $P_\alpha = C_\alpha^{\frac{1}{2}}$. We shall find that

$$(1.2)\qquad \begin{aligned} C_\alpha^\nu(\cos\theta) &= \tfrac{1}{2}A_0 + \Sigma_{n=1}^{\infty} A_n \cos n\theta\ , \qquad (\mathrm{Re}\,\nu < 1),\\ A_n &= \left\{1 + \frac{\sin\pi(\nu+\alpha+n)}{\sin\pi\nu}\right\} \frac{\Gamma(\nu+\frac{\alpha+n}{2})\,\Gamma(\nu+\frac{\alpha-n}{2})}{[\Gamma(\nu)]^2\,\Gamma(1+\frac{\alpha+n}{2})\,\Gamma(1+\frac{\alpha-n}{2})}\ . \end{aligned}$$

If Re $\nu \geq 1$ the Fourier coefficients do not exist for general α because $C_\alpha^\nu(\cos\theta)$ is not integrable over an interval containing the point $\theta = \pi$. If Re $\nu < 1$ and α is a positive integer N, the first factor of A_n vanishe if $N + n$ is odd; since $A_{-n} = A_n$, (1.2) then reduces to (1.1).

We remark that Gegenbauer's function (multiplied by a constant to

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give it the value unity at $\theta = 0$) has a nicely symmetrical expression in the notation of the hypergeometric R function [1]:

$$(1.3)\qquad \frac{\Gamma(2\nu)\,\Gamma(\alpha+1)}{\Gamma(2\nu+\alpha)}\,C_\alpha^\nu(\cos\theta) = R(-\alpha;\nu,\nu;e^{i\theta},e^{-i\theta})$$

$$= 2^{\nu-\frac{1}{2}}\,\Gamma(\nu+\tfrac{1}{2})\,(\sin\theta)^{\frac{1}{2}-\nu}\,P^{\frac{1}{2}-\nu}_{\alpha+\nu-\frac{1}{2}}(\cos\theta)\ ,$$

where P is an associated Legendre function. An important special case is

$$(1.4)\qquad C_\alpha^{\frac{1}{2}}(\cos\theta) = R(-\alpha;\tfrac{1}{2},\tfrac{1}{2};e^{i\theta},e^{-i\theta}) = P_\alpha(\cos\theta)\ .$$

2. The Fourier coefficients. Gegenbauer's function is defined [3, vol. 1, p. 178] by

$$(2.1)\qquad \frac{\Gamma(2\nu)\,\Gamma(\alpha+1)}{\Gamma(2\nu+\alpha)}\,C_\alpha^\nu(\cos\theta) = {}_2F_1(-\alpha,2\nu+\alpha;\nu+\tfrac{1}{2};\sin^2\tfrac{\theta}{2})$$

$$= \sum_{m=0}^{\infty}\frac{(-\alpha)_m\,(2\nu+\alpha)_m}{(\nu+\frac{1}{2})_m\ m!}\,\sin^{2m}(\theta/2)\ .$$

We assume for the present that $\operatorname{Re}\nu < \frac{1}{2}$, so that the hypergeometric series converges absolutely for $\theta = \pi$ [5, p. 25] and hence uniformly over the interval $(0, \pi)$. The Fourier coefficient

$$(2.2)\qquad A_n = \frac{2}{\pi}\int_0^\pi C_\alpha^\nu(\cos\theta)\cos n\theta\,d\theta$$

can then be found by integrating term by term. From the elementary formula

$$(2.3)\qquad \frac{2}{\pi}\int_0^\pi \sin^{2m}(\theta/2)\cos n\theta\,d\theta = (-1)^n\,2^{1-2m}\binom{2m}{m+n}\ ,$$

we get

$$A_n = \frac{(-1)^n \, 2\,\Gamma(2\nu+\alpha)}{\Gamma(2\nu)\,\Gamma(\alpha+1)} \sum_{m=n}^{\infty} \frac{(-\alpha)_m \, (2\nu+\alpha)_m \, (\frac{1}{2})_m}{(\nu+\frac{1}{2})_m \, (m+n)! \, (m-n)!}$$

(2.4)

$$= \frac{(-1)^n \, 2^{1-2n} \, \Gamma(2\nu+\alpha+n) \, (-\alpha)_n}{\Gamma(2\nu)\,\Gamma(\alpha+1)\,(\nu+\frac{1}{2})_n \; n!} \sum_{k=0}^{\infty} \frac{(n-\alpha)_k \, (2\nu+\alpha+n)_k \, (\frac{1}{2}+n)_k}{(\nu+\frac{1}{2}+n)_k \, (1+2n)_k \, k!} .$$

The last series, obtained from the preceding one by putting $m = n + k$, is a ${}_3F_2$ series with unit argument. If Re $\nu < 1$ it converges and can be summed by Watson's theorem [3, vol. 1, p. 189]:

$$(2.5) \quad {}_3F_2(n-\alpha,\, 2\nu+\alpha+n,\, \tfrac{1}{2}+n;\, \nu+\tfrac{1}{2}+n,\, 1+2n;\, 1)$$

$$= \frac{\pi^{\frac{1}{2}} \, n! \, \Gamma(\nu+\frac{1}{2}+n)\,\Gamma(1-\nu)}{\Gamma(\frac{1+n-\alpha}{2})\,\Gamma(\frac{1+n+2\nu+\alpha}{2})\,\Gamma(1+\frac{n+\alpha}{2})\,\Gamma(1+\frac{n-2\nu-\alpha}{2})} .$$

Substitution in (2.4) gives an expression for A_n that can be simplified by applying several times the duplication formula for the gamma function and the relation $\Gamma(z)\,\Gamma(1-z) = \pi \csc \pi z$. The result is

$$(2.6) \quad A_n = \frac{(-1)^n \sin \pi\alpha \;\; \sin \pi(\nu + \frac{\alpha-n}{2})\,\Gamma(\nu+\frac{\alpha+n}{2})\,\Gamma(\nu+\frac{\alpha-n}{2})}{\sin \pi\nu \;\; \sin \pi(\frac{\alpha-n}{2})\,[\Gamma(\nu)]^2\,\Gamma(1+\frac{\alpha+n}{2})\,\Gamma(1+\frac{\alpha-n}{2})} .$$

Elementary rearrangement of the sine functions now leads to (1.2).

If Re $\nu \geq \frac{1}{2}$ the series (2.1) no longer converges uniformly over $(0, \pi)$ if it does not terminate. However, the analytic continuation of Gauss' hypergeometric function [5, p. 291] shows that $C_\alpha^\nu(\cos\theta)$ is then of the order of $(\cos \frac{1}{2}\theta)^{1-2\nu}$ as $\theta \to \pi$ (except that the singularity is logarithmic if $\nu = \frac{1}{2}$).

Provided that Re $\nu < 1$, the function is integrable over $(0, \pi)$; furthermore, it satisfies conditions [5, p. 164] sufficient to ensure that its Fourier series converges (except when θ is an odd multiple of π) and represents the function.

To show that the Fourier coefficients are still given by (1.2) if $\frac{1}{2} \leq$ Re $\nu < 1$, one can either use analytic continuation in ν or justify directly the term-by-term integration of (2.1). The second method is the easier if one uses the following theorem [4, p. 45]: If $\Sigma\, u_m(\theta)$ converges uniformly over $(0, \pi - \epsilon)$ for every (small) positive ϵ, and if $\Sigma \int_0^\pi |u_m(\theta)|\, d\theta$ converges, then $\Sigma\, u_m(\theta)$ may be integrated term by term over $(0, \pi)$.

The first assumption of the theorem is plainly satisfied by

$$u_m(\theta) = \frac{(-\alpha)_m\, (2\nu + \alpha)_m}{(\nu + \frac{1}{2})_m\, m!} \sin^{2m}(\theta/2) \cos n\theta \quad .$$

Moreover, we have

$$\Sigma \int_0^\pi |u_m(\theta)|\, d\theta \;\leq\; \Sigma \left| \frac{(-\alpha)_m\, (2\nu + \alpha)_m}{(\nu + \frac{1}{2})_m\, m!} \right| \int_0^\pi \sin^{2m}(\theta/2)\, d\theta$$

$$= \pi\, \Sigma \left| \frac{(-\alpha)_m\, (2\nu + \alpha)_m\, (\frac{1}{2})_m}{(\nu + \frac{1}{2})_m\, m!\, m!} \right| .$$

The last series converges if Re $\nu < 1$, and the proof of (1.2) is now complete.

Eq. (1.3) follows from (2.1) by use of the relation [1, Eq. (2.5)]

(2.7) $$ {}_2F_1(a, b; c; x) = R(a; b, c - b; 1 - x, 1)$$

and the quadratic transformation [2, Eq. (5.1)]

(2.8) $$R(a; b, b; x^2, y^2) = R(a; 2b - a, a - b + \tfrac{1}{2}; (x + y)^2/4, xy) \; .$$

References

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